VINNITSA NATIONAL AGRARIAN UNIVERSITY

Department of General Engineering Sciences and Labour Safety

CALCULATION OF TRANSIENTS IN ELECTRICAL CIRCUITS OF THE SECOND ORDER

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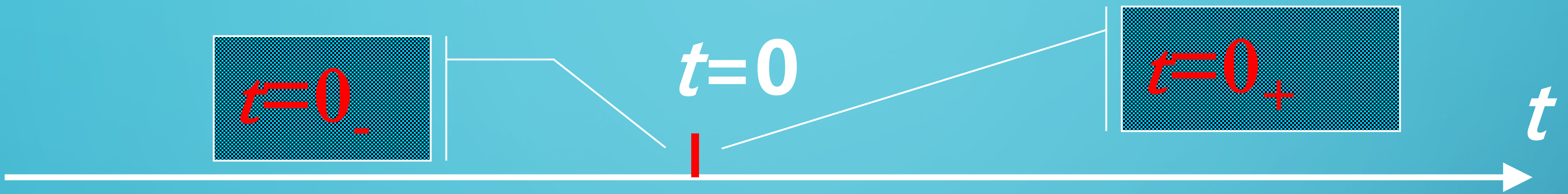
ALGORITHM FOR CALCULATING TRANSIENTS IN COMPLEX ELECTRICAL CIRCUITS

1. Compose a system of equations for an electric circuit according to Kirchhoff's rules in an instantaneous form.
2. Based on the system of equations to obtain an inhomogeneous differential equation.
3. Based on the inhomogeneous differential equation to obtain the characteristic equation. Find its solution.
4. Find homogeneous solution (own component).
5. Find particular solution (forced component).

RULES OF COMMUTATION



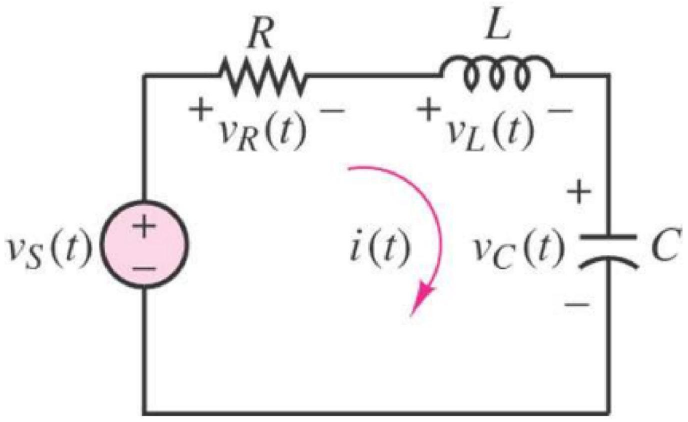
$$\begin{cases} W_L(0_-) = W_L(0_+) \\ W_C(0_-) = W_C(0_+) \end{cases}$$

$$\begin{cases} i_L(0-) = i_L(0+) \\ u_C(0-) = u_C(0+) \end{cases}$$

EXAMPLE 1 OF OBTAINING A DIFFERENTIAL EQUATION

Kirchhoff's first rule $: i_R = i_C = i_L = i$

Kirchhoff's second rule $: -v_S + v_R + v_C + v_L = 0 \rightarrow -v_S + i_R R + v_C + v_L = 0$



$$iR + v_C(t=0) + \int_0^t \frac{i(t')}{C}dt' + L\frac{di}{dt} = v_S$$

$$\frac{di}{dt}R + \frac{i}{C} + L\frac{d^2i}{dt^2} = \frac{dv_S}{dt} \rightarrow \frac{d^2i}{dt^2} + \frac{Rdi}{Ldt} + \frac{i}{LC} = \frac{dv_S}{Ldt} : \text{Differential equation for } i$$

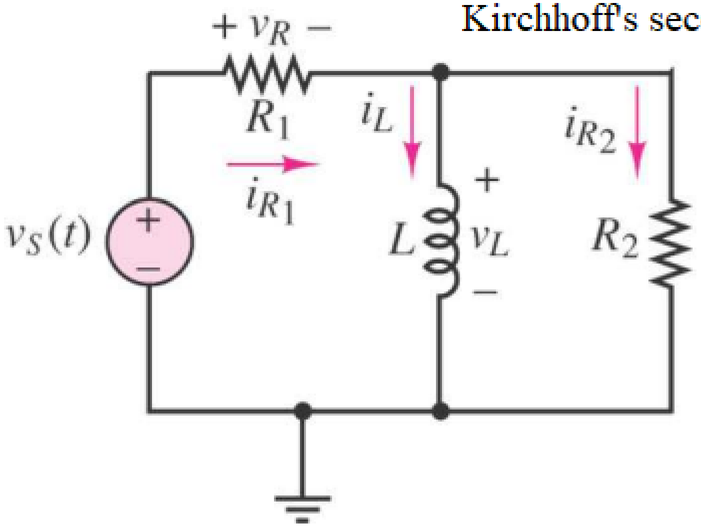
$$\frac{v_R}{R} = i_C = C\frac{dv_C}{dt} = \frac{v_S - v_C - v_L}{R} \rightarrow C\frac{dv_C}{dt} = \frac{v_S}{R} - \frac{v_C}{R} - \frac{1}{R}\left(L\frac{d}{dt}C\frac{dv_C}{dt}\right)$$

$$RC\frac{dv_C}{dt} = v_S - v_C - LC\left(\frac{d^2v_C}{dt^2}\right)$$

$$LC\left(\frac{d^2v_C}{dt^2}\right) + RC\frac{dv_C}{dt} + v_C = v_S : \text{Differential equation for } v_C$$

$$a_2\frac{d^2x(t)}{dt^2} + a_1\frac{dx(t)}{dt} + a_0x(t) = b_0f(t) \rightarrow \text{Second - order linear ordinary differential equation}$$

EXAMPLE 2 OF OBTAINING A DIFFERENTIAL EQUATION



Kirchhoff's first rule : $i_{R_1} = i_L + i_{R_2} \rightarrow \frac{v_R}{R} = i_L + i_{R_2}$

Kirchhoff's second rule : $-v_S + v_R + v_L = 0 \rightarrow v_R = v_S - v_L$

$$\frac{v_R}{R} = i_L + i_{R_2} \rightarrow \frac{v_S - v_L}{R} = i_L(t=0) + \int_0^t \frac{v_L(t')}{L} dt' + \frac{v_L}{R}$$

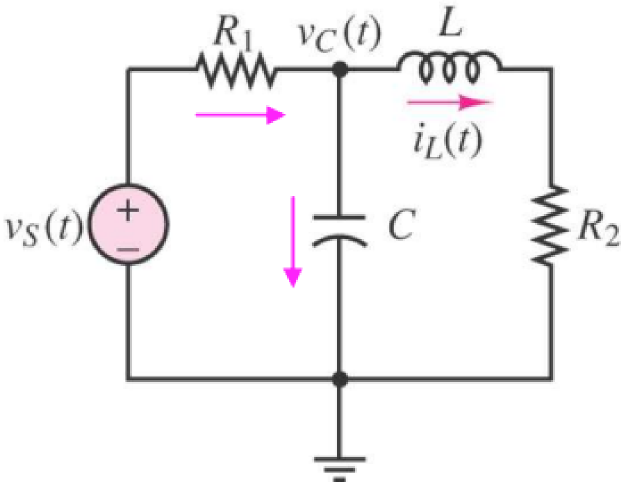
$$v_S - v_L = Ri_L(t=0) + \int_0^t \frac{Rv_L(t')}{L} dt' + v_L \rightarrow v_S = Ri_L(t=0) + \int_0^t \frac{Rv_L(t')}{L} dt' + 2v_L$$

$$\frac{dv_S}{dt} = \frac{R}{L} v_L + \frac{2dv_L}{dt} \rightarrow 2\frac{dv_L}{dt} + \frac{R}{L} v_L = \frac{dv_S}{dt}$$: Differential equation for v_L

EXAMPLE 3 OF OBTAINING A DIFFERENTIAL EQUATION

Kirchhoff's first rule : $i_{R_1} = i_C + i_L$

Kirchhoff's second rule : $-v_S + v_{R_1} + v_C = 0 \rightarrow v_S = v_{R_1} + v_C$



$$-v_C + v_{R_2} + v_L = 0 \rightarrow v_C = v_{R_2} + v_L = L\frac{di_L}{dt} + i_L R_2$$

$$v_{R_1} = i_{R_1} R_1 = (i_C + i_L) R_1 = \left(C\frac{dv_C}{dt} + i_L\right) R_1 = \left(C\frac{d}{dt}\left(L\frac{di_L}{dt} + i_L R_2\right) + i_L\right) R_1 = \left(LC\frac{d^2 i_L}{dt^2} + R_2 C\frac{di_L}{dt} + i_L\right) R_1$$

$$v_S = v_{R_1} + v_C = \left(LC\frac{d^2 i_L}{dt^2} + R_2 C\frac{di_L}{dt} + i_L\right) R_1 + L\frac{di_L}{dt} + i_L R_2 \rightarrow$$

$$v_S = R_1 LC\frac{d^2 i_L}{dt^2} + R_1 R_2 C\frac{di_L}{dt} + R_1 i_L + L\frac{di_L}{dt} + i_L R_2 \rightarrow$$

$$R_1 LC\frac{d^2 i_L}{dt^2} + (R_1 R_2 C + L)\frac{di_L}{dt} + (R_1 + R_2) i_L = v_S$$: Differential equation for i_L

HOMOGENEOUS SOLUTION OF SECOND-ORDER DIFFERENTIAL EQUATION

characteristic equation

$$a \cdot p^2 + b \cdot p + c = 0$$

Depending on the ratio of components under the sign of the radical, we will have three types of solutions (roots).

$b^2 < 4 \cdot a \cdot c$ **- complex-conjugate roots;**

$b^2 > 4 \cdot a \cdot c$ **- roots are real and different;**

$b^2 = 4 \cdot a \cdot c$ **- roots are real and the same.**

COMPLEX-CONJUGATE ROOTS

The roots can be represented as: $p_1 = -\delta + j\omega_0 \qquad p_2 = -\delta - j\omega_0$

The transient process of the circuit will be periodic (oscillating)

For example, the transient voltage at the capacitor can be written as

$$u_C(t) = e^{-\delta t}\left(A_1 \sin\omega_0 t + A_2 \cos\omega_0 t\right)$$

ROOTS ARE REAL AND DIFFERENT

The transient process will be aperiodic (non-oscillating)

Then the transient voltage on the capacitor can be written as

$$u_C(t) = A_1 e^{p_1 t} + A_2 e^{p_2 t}$$

ROOTS ARE REAL AND THE SAME

The transient process is critical. It is a transition between aperiodic and oscillatory processes

In this case, the voltage on the capacitor is written as

$$u_C = \left(A_1 + A_2 t\right) e^{pt}$$

EXAMPLE 1 OF CALCULATING THE TRANSIENT PROCESS IN A BRANCHED CIRCLE OF THE SECOND ORDER

Step1: $i_S = i_C = i_L \rightarrow \frac{v_R}{R_T} = i_L + i_C$

$-v_S + v_R + v_L + v_C = 0 \rightarrow v_R + v_L + v_C = v_S$

$i_L R + L\frac{di_L}{dt} + v_C(t=0) + \int_0^t \frac{i_L(t')}{C}dt' = v_S \rightarrow L\frac{d^2 i_L}{dt^2} + R\frac{di_L}{dt} + \frac{i_L}{C} = \frac{dv_S}{dt} = 0$

Step2: $v_C(t=0^-) = 5\text{ V} = v_C(t=0^+), i_L(t=0^-) = 0\text{ A} = i_L(t=0^+)$

$i_L(t=0^+)R + L\frac{di_L}{dt}(t=0^+) + v_C(t=0) = v_S \rightarrow 1\frac{di_L}{dt}(t=0^+) + 5\text{ V} = 25\text{V} \rightarrow \frac{di_L}{dt}(t=0^+) = 20\text{A/s}$

Step3: $L\frac{d^2 i_L}{dt^2} + R\frac{di_L}{dt} + \frac{i_L}{C} = 0 \rightarrow LC\frac{d^2 i_L}{dt^2} + RC\frac{di_L}{dt} + i_L = 0 : \frac{1}{\omega_n^2}\frac{d^2 x(t)}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dx(t)}{dt} + x(t) = K_S f(t)$

$\frac{1}{\omega_n^2} = LC \rightarrow \omega_n = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{10^{-6}}} = 1000\text{ (rad/s)}, \frac{2\zeta}{\omega_n} = RC \rightarrow \zeta = \frac{RC\omega_n}{2} = \frac{R}{2}\sqrt{\frac{C}{L}} = \frac{5000}{2}\sqrt{\frac{10^{-6}}{1}} = 2.5$

$\rightarrow$ Overdamped response

$i_L(t) = \alpha_1 e^{s_1 t} + \alpha_2 e^{s_2 t}$ where $s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$

Complete Response (forced response = 0)

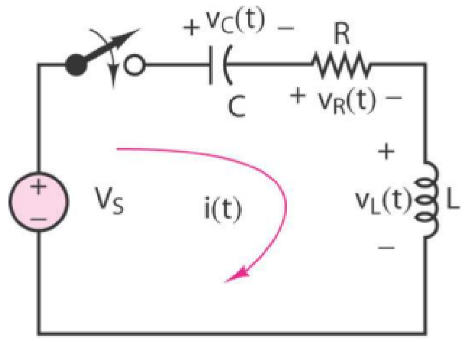
$i_L(t) = \alpha_1 e^{\left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)t} + \alpha_2 e^{\left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)t}$

Step4: Using $0\text{ A} = i_L(t=0^+)$ and $\frac{di_L}{dt}(t=0^+) = 20\text{A/s}$, determine the constants α_1 and α_2

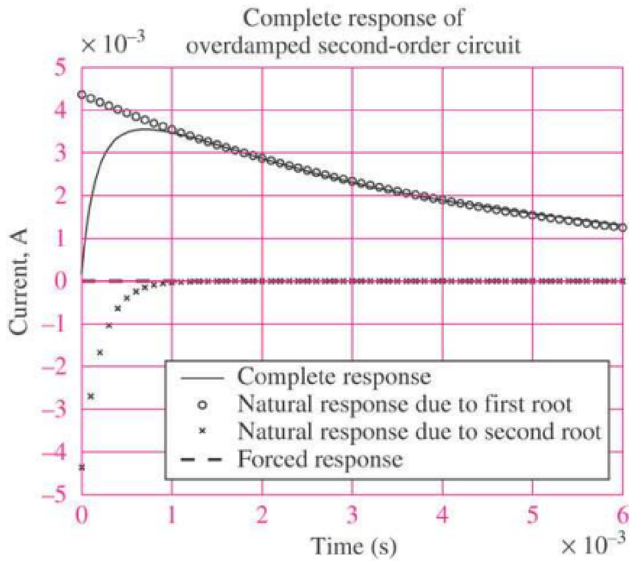
$i_L(t=0^+) = 0 = \alpha_1 + \alpha_2$

$\frac{di_L}{dt} = \alpha_1\left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)e^{\left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)t} + \alpha_2\left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)e^{\left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)t}$

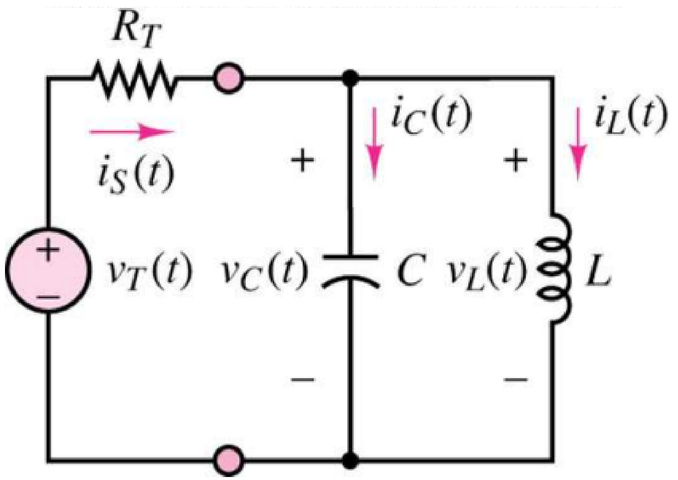
$\frac{di_L}{dt}(t=0^+) = 20 = \alpha_1\left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right) + \alpha_2\left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)$



R = 5,000 Ω L = 1 H C = 1 μF
VS = 25 V



EXAMPLE 2 OF CALCULATING THE TRANSIENT PROCESS IN A BRANCHED CIRCLE OF THE SECOND ORDER



$$i_S = i_C + i_L \rightarrow \frac{v_R}{R_T} = i_L + i_C$$

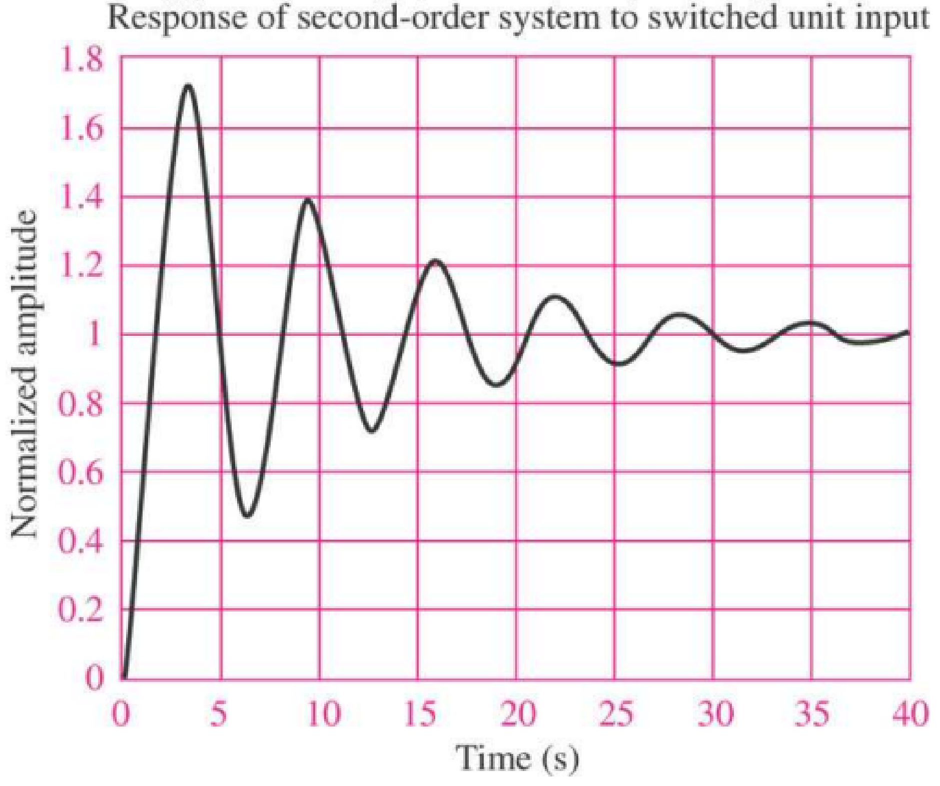
$$-v_T + v_R + v_L = 0 \rightarrow v_R = v_T - v_L \text{ and } -v_T + v_R + v_C = 0 \rightarrow v_R = v_T - v_C$$

$$\frac{v_R}{R_T} = i_L + i_C \rightarrow \frac{1}{R_T}\left(v_T - L\frac{di_L}{dt}\right) = i_L + C\frac{dv_C}{dt} = i_L + C\frac{d}{dt}\left(L\frac{di_L}{dt}\right)$$

$$\frac{1}{R_T}\left(v_T - L\frac{di_L}{dt}\right) = i_L + LC\frac{d^2 i_L}{dt^2} \rightarrow \frac{v_T}{R_T} = LC\frac{d^2 i_L}{dt^2} + \frac{L}{R_T}\frac{di_L}{dt} + i_L$$

$$a_2 \frac{d^2x(t)}{dt^2} + a_1 \frac{dx(t)}{dt} + a_0 x(t) = b_0 f(t) \rightarrow \frac{1}{\omega_n^2} \frac{d^2x(t)}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx(t)}{dt} + x(t) = K_S f(t)$$

where the constants $\omega_n = \sqrt{a_0/a_2}$, $\zeta = (a_1/2)\sqrt{1/a_0 a_2}$ and $K_S = b_0/a_0$ termed the natural frequency, the damping ratio, and the DC gain, respectively.



$\omega_n = 1, \zeta = 0.1$ and $K_S = 1$

- The final value of 1 is predicted by the DC gain K_S=1, which tells us about the steady state.
- The period of oscillation of the response is related to the natural frequency w_n=1 leads to T=2 pi/w_n = 6.28 sec.
- The reduction in amplitude of the oscillation is governed by the damping ratio. With large damping ratio, the response not overshoots (oscillates) but looks like the first order response.
- Damping -> friction effect

$$\frac{1}{\omega_n^2}\frac{d^2x(t)}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dx(t)}{dt}+x(t)=K_S f(t)$$

Natural Response

$$\frac{1}{\omega_n^2}\frac{d^2x_N(t)}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dx_N(t)}{dt}+x_N(t)=0$$

$x_N(t)=\alpha_1 e^{s_1 t}+\alpha_2 e^{s_2 t}$ where $s_{1,2}=-\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$

Case 1 : Real and distinct roots. $(\zeta>1)\rightarrow$ Overdamped response

$\rightarrow$ Look like the first order system

$s_{1,2}=-\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$

Case 2 : Real and repeated roots. $(\zeta=1)$

$\rightarrow$ Critically overdamped response $\rightarrow$ Oscillation

$s_{1,2}=-\omega_n$

Case 3 : Complex roots. $(\zeta<1)\rightarrow$ Underdamped response $\rightarrow$ Oscillation

$s_{1,2}=-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$

Forced Response due to DC (where $f(t)=F$): $\frac{dx_F(t)}{dt}\rightarrow 0$

$$\frac{1}{\omega_n^2}\frac{d^2x_F(t)}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dx_F(t)}{dt}+x_F(t)=K_S f(t)\ \ t\geq 0 \rightarrow x_F(t)=K_S F\ \ t\geq 0$$

Complete Response

$x(t)=x_N(t)+x_F(t)$ α_1 and α_2 is constants that will be determined by the initial conditions.